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ON THE STABILITY OF A STATIONARY FRONT OF AN EXOTHERMIC REACTION IN CYLINDRICAL SAMPLES*

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The problem of the stability of a planar combustion wave front which propagates in a thermally isolated porous cylinder is considered. A model system of the theory of combustion is used which describes an exothermic reaction in a porous medium saturated with gas. The Arrhenius dependence of the reaction rate is replaced by a piecewise-constant dependence. In this case it is possible to find an analytical solution for the temperature distribution and the distribution of the reagent concentration in the stationary wave. Unlike the results obtained using the approach in /1, 2/, the thermal and diffusion fluxes are continuous everywhere in the case of the solution which is constructed. Investigation of the stability of the solution leads to results which are similar to the approximate results in /1, 2/ but the details concerning loss of stability exhibit a qualitative difference associated with the occurrence of folding in the neighbourhood of the neutral hypersurface at a Lewis number $L<1$ which accounts for the possible appearance of vortex waves in a sample of circular cross-section /3, 4/.

When $0<L<1$ on the neutral surface, the boundary of stability of a stationary combustion front in the cylindrical sample is determined from data on the vibrational frequency spectrum of a free membrane which is identical in shape with the cross-section of the cylindrical sample. This boundary depends on the shape of the cross-section of the sample but is always enclosed between the stability boundary of the wave in unbounded space and the stability boundary in the corresponding one-dimensional problem. Loss of stability occurs when a pair of complex conjungate eigenvalues intersect the imaginary axis. When $L>1$, loss of stability occurs when a real eigenvalue intersects the imaginary axis. The region of stability for a cylinder of any cross-sectional form is wider than the stability region which corresponds to a wave propagating in an unbounded space filled with a substance. The critical indices of the stability boundary in the neighbourhood of $L=1$ are determined together with the critical dimensions for samples of square and circular cross-section.

The thermal conductivity and diffusion equations in a coordinate system which moves at constant velocity U along the generatrix of a cylinder have the form

$$\frac{\partial X}{\partial \tau}=\frac{\partial^2 X}{\partial x^2}+\frac{\partial^2 X}{\partial y^2}+\frac{\partial^2 X}{\partial s^2}+\frac{\partial X}{\partial s}-\Phi \tag{1}$$

$$\frac{\partial Y}{\partial \tau}=L\left(\frac{\partial^2 Y}{\partial x^2}+\frac{\partial^2 Y}{\partial y^2}+\frac{\partial^2 Y}{\partial s^2}\right)+\frac{\partial Y}{\partial s}-\Phi$$

$$\Phi=\Lambda Y f(X,\theta),\quad f(X,\theta)=\exp\left(\frac{1}{\theta}\frac{X}{X-1}\right) \tag{2}$$

$$\tau=\frac{U^2}{a}t,\quad s=\frac{U}{a}(z_3-Ut),\quad x=\frac{U}{a}z_1,\quad y=\frac{U}{a}z_2$$

$$X=1-\frac{T}{T_1},\quad Y=\frac{q}{\theta}n,\quad T_1=T_0+\frac{Q}{c}$$

$$L=\frac{D}{a},\quad q=\frac{RQ}{Ec},\quad \theta=\frac{RT_1}{E},\quad \Lambda=\frac{aB}{U^2}\exp\left(-\frac{1}{\theta}\right)$$

Here z_1, z_2, z_3 are Cartesian coordinates, Φ is the reaction rate, n is the concentration of the reagent, D and a are the coefficient of diffusion and the thermal conductivity, B is a pre-exponential factor, T_0 is the initial temperature and T_1 is the combustion temperature. The remaining notation is generally accepted /5/. The quantities L, θ and q $(0<q<\theta)$ are the dimensionless parameters of the problem which correspond to the coefficient of diffusion, the combustion temperature and the amount of heat evolved. The dependent parameter $C=q/\theta$ is introduced.

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In the problem under consideration, as in /6/, the Arrhenius dependence of the reaction rate on temperature in (1) is replaced by the model dependence (h is the Heaviside function)

$$f(X, \theta) = h(\theta - X) \tag{3}$$

For Eqs.(1) in the case of (2) or (3), the initial-boundary value problem (Ω is the cylinder and ν is the external normal to the boundary of the cylinder) becomes

$$X|_{\tau=0} = Y_0(x, y, s), \quad Y|_{\tau=0} = X_0(x, y, s), \quad \frac{\partial X}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial Y}{\partial \nu}\Big|_{\partial\Omega} = 0$$

Let U be identical with the velocity of propagation of the combustion wave. The profile of this wave is then determined by the stationary solution of (1) with the boundary conditions

$$s = -\infty, \quad X = Y = 0, \quad s = +\infty, \quad X = Y = C$$

The stationary problem leads to the system

$$\frac{dX}{ds} = Z - X, \quad L\frac{dY}{ds} = Z - Y, \quad \frac{dZ}{ds} = \Lambda Y f(X, \theta) \tag{4}$$
$$s = -\infty, \quad X = Y = Z = 0, \quad s = +\infty, \quad X = Y = Z = C$$

The second of Eqs.(4) is obtained by subtracting the second equation of (1) from the first and integrating with respect to s taking account of the boundary conditions. The velocity of the wave is determined by the dependence found by solving (4) which relates Λ, L, C, and θ. In the case of (3), the solution has the form

$$\begin{aligned} &X = \exp(ps), \quad Y = p(1+p)\Lambda^{-1}\exp(ps), \quad s \leqslant s_* \\ &X = C - \exp(s_1 - s), \quad Y = C - \exp((s_2 - s)/L), \quad s > s_* \\ &s_* = p^{-1}\ln\theta, \quad s_1 = s_* + \ln(p\theta), \quad s_2 = s_* + L\ln[CLp/(1+Lp)^{-1}] \\ &C = (1+p)\theta, \quad p = ((1 + 4L\Lambda)^{1/2} - 1)/(2L) \end{aligned} \tag{5}$$

Here s_1 is the width of the thermal wave and s_2 is the width of the conversion wave. By means of the transformation $X = \theta X'$, $Y = \theta Y'$, Eqs.(1) can be reduced to a form which only contains the parameter Λ while the boundary conditions of the stationary problem (4) only contain the parameter C/θ or the parameter p which is associated with it by virtue of (5). The stationary problem therefore only depends on Λ and L or on p and L. We shall subsequently take p and L as the fundamental parameters of the problem. As $L \to 0$, the parameter p transforms into Λ.

The asymptotic behaviour of the stationary solution when $s \to \pm\infty$ in the case of (2) has the same form in the case of (3).

Let us now investigate the stability of the solution (5) within the framework of linear theory. The eigenvalue problem has the form

$$\begin{aligned} &\lambda u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial s^2} + \frac{\partial u}{\partial s} - \frac{\partial \Phi}{\partial X}u - \frac{\partial \Phi}{\partial Y}v \\ &\lambda v = L\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial s^2}\right) + \frac{\partial v}{\partial s} - \frac{\partial \Phi}{\partial X}u - \frac{\partial \Phi}{\partial Y}v \\ &u, v \to 0, \quad s \to \pm\infty, \quad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0 \end{aligned} \tag{6}$$

where $\partial\Phi/\partial X$ and $\partial\Phi/\partial Y$ are calculated on the solution (5)

$$\partial\Phi/\partial X = -(1+p)\,\delta(s - s_*), \quad \partial\Phi/\partial Y = \Lambda h(s_* - s)$$

By applying the method of the separation of variables to (6)

$$u = \xi(s)\,w(x, y), \quad v = \eta(s)\,w(x, y) \tag{7}$$

we obtain two problems:

$$\begin{aligned} &\frac{d\xi}{ds} = \zeta, \quad \frac{d\eta}{ds} = \rho, \quad \frac{d\zeta}{ds} = \left(\frac{\partial \Phi}{\partial X} + \lambda + \mu\right)\xi + \frac{\partial \Phi}{\partial Y}\eta - \zeta \\ &L\frac{d\rho}{ds} = \frac{\partial \Phi}{\partial X}\xi + \left(\frac{\partial \Phi}{\partial Y} + \lambda + L\mu\right)\eta - \rho \\ &\xi, \eta, \zeta, \rho \to 0, \quad s \to \pm\infty \end{aligned} \tag{8}$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \mu w = 0, \quad \frac{\partial w}{\partial \nu}\Big|_{\partial\Omega} = 0 \tag{9}$$

We note that (9) is the problem of the vibration of a free membrane which has a denumerable number of non-negative solutions μ_n ($\mu_0 = 0$) which are ordered in magnitude and to each of which several eigenfunctions w may correspond. The characteristic numbers, λ, are determined for each μ_n from (8) while the eigenfunctions (6), corresponding to the given values of λ are found using formula (7). In order to determine the boundary of stability of a cylinder with a cross-section Ω one merely needs to know the spectrum $\mu_n(\Omega)$, $n = 0, 1, \ldots$, by which the shape Ω is not uniquely determined. The solution of the stability problem involves the determination of the boundary of stability of the stationary wave in the space of the parameters $L, p, \{\mu_n\}$ from the condition of intersection of the imaginary axis by the right-most eigenvalue.

Problem (8) depends on problem (9) only through the parameter μ and (8) can therefore be considered as a single parameter problem in μ on the eigenvalues λ. When $\mu = 0$, this is the problem of the one-dimensional stability of a stationary wave. Problem (8) may also be considered as a problem of the stability of a stationary wave which propagates in an unbounded space filled with a substance. In fact, by applying a Fourier transform with respect to x and y to (6) and introducing the notation (U and V are the Fourier transforms of the functions u, v)

$$\xi = U, \quad \eta = V, \quad \zeta = \partial U/\partial s, \quad \rho = \partial V/\partial s, \quad \mu = k_x^2 + k_y^2$$

we obtain (8). Moreover, the perturbations of the basic solution (5) have the form

$$\begin{pmatrix}\delta X\\ \delta Y\end{pmatrix} = \operatorname{Re}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\exp(\lambda\tau + ik_x x + ik_y y)$$

Hence, yet another parameter μ appears in problem (8) in addition to the basic parameters of the problem L and p. In the space of L, p, μ (or L, Λ, μ), where $L \geqslant 0$, $\mu \geqslant 0$, $p > 0$ ($\Lambda > 0$), the condition for the stability of the stationary wave determines the neutral surface $\Lambda = \Lambda(\mu, L)$ along which the stability boundary will be determined. In the region where $\Lambda > \Lambda(\mu, L)$, λ are the eigenvalues in the right-hand half-plane. Sections of the surface $\Lambda = \Lambda(\mu, L)$ with the planes $L = 0.1N$, where $N = 0, 1, \ldots, 6$ is the number of the neutral curve are shown in Fig.1. When $L > 1$, the cross-sections of the surface $p(\mu, L)$ which corresponds to $\Lambda(\mu, L)$ by virtue of (5) are shown in Fig.2 for $\mu = N$, where $N = 0, 2, 4$,

The determination of the neutral curve $\Lambda = \Lambda(\mu, L)$ when $L = 0$ requires that certain changes are made in the formulation of the problem since the order of Eqs.(1) is changed when this is done. However, as a study of this case has shown, the equation for determining the neutral curve when $L = 0$ is identical to the equation which is obtained from the general case when $L \to 0$ and we shall therefore subsequently assume that $L > 0$.

When $0 \leqslant L < 1$, all the neutral curves intially decrease from the value $\Lambda^*(L)$ to $\Lambda_*(L)$ (when $\mu = \mu_*(L)$) and then increase without limit (Fig.1). When the neutral surface is intersected in the plane of λ, a pair of complex conjugate eigenvalues intersect the imaginary axis and the values $\lambda = \pm i\omega$ determine the function on the neutral surface $\omega = \omega(k, L)$, where $k = \sqrt{\mu}$. Profiles of this dependence when $L = 0{,}2N$, where $N = 0, 1, 3, 5$ are presented in Fig.3. The last of these profiles is to be considered as the limiting profile as $L \to 1 - 0$. The profiles are readily distinguished by the values of ω at $k = 0$. The dependence $\omega^*(L)$ decreases monotonically as L increases (Fig.4) and determines the frequency of the autovibrations when there is loss of stability in the one-dimensional problem. The value of $\Lambda_*(L)$ determines the stability boundary of a stationary wave which propagates in an unbounded medium while $\Lambda^*(L)$ determines the boundary of one-dimensional stability of a stationary wave.

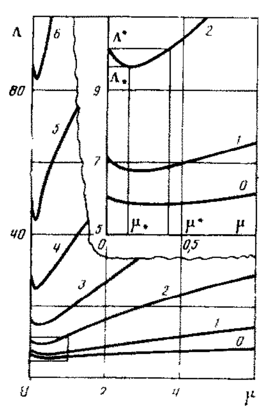

Fig.1

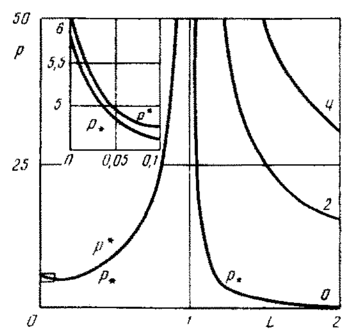

Fig.2

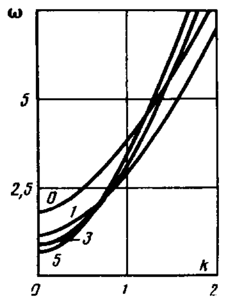

Fig.3

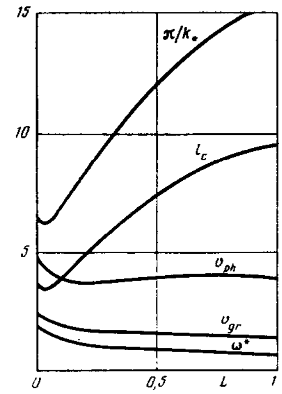

Fig.4

In the case of a cylinder of cross-section Ω the boundary of stability is determined from the relationship

$$\Lambda_\Omega(L) = \min_n \Lambda(\mu_n, L) \tag{10}$$

and is always included between $\Lambda_*(L)$ and $\Lambda^*(L)$. Using (5), these same boundaries $p_*(L)$ and $p^*(L)$ may be considered in the p, L plane. The upper and lower stability bounds almost merge (Fig.2) and have the asymptotic forms $p^* \approx 5.46/(1-L) - 2.7$, $p_* \approx 5{,}33/(1-L) - 2.6$ as $L \to 1-0$.

The exact values of the coefficients for negative moduli of $(1-L)$ are equal to $2(1+\sqrt{3})$ and 16/3 respectively. The lower stability boundary $\Lambda_*(L)$ has a minimum value $\Lambda_{**} \approx 5.790$ at $L \approx 0.0137$. The value of n, at which a minimum is attained in (10), determines the most "dangerous" perturbation modes, the half-wave lengths π/k and the phase and group velocities of these perturbations $v_{ph}(L) = \omega/k$ and $v_{gr}(L) = d\omega/dk$ when $k = k_n(L), k_n = \sqrt{\mu_n}$. In the calculation of these quantities in the case of a wave which propagates in an unbounded space, it follows that one should put $k = k_*(L)$ and $k_* = \sqrt{\mu_*(L)}$ (Fig.4).

When $L > 1$, loss of stability occurs where the imaginary axis is intersected by a real eigenvalue λ and, since $\Lambda(\mu, L)$ increases as μ increases, the minimum in (10) is attained when $n = 1$, that is, $\Lambda_\Omega(L) = \Lambda(\mu_1, L)$. Here the value $\mu_0 = 0$ has not been taken into account since, here, neutral perturbations which solely lead to a displacement of the wave front along the s-axis (see /6/) correspond to the eigenvalue $\lambda = 0$. This limits of stability in the variables L and p when $\mu_1 = 2.4$ are presented in Fig.2. When $p > p(\mu_1, L)$, the stationary wave is unstable.

Hence, when $L > 1$, the stability boundary is represented by a surface in the space of the parameters $L, p,$ and μ_1 and does not depend on the values of the other μ_n. In the case, the perturbation of the main solution (5) is

$$\begin{pmatrix}\delta X\\ \delta Y\end{pmatrix} = \operatorname{Re}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\exp(\lambda\tau)\, w_1(x, y) \tag{11}$$

where $\xi(s)$ and $\eta(s)$ are solutions of problem (8) when $\lambda = 0$ and $\mu = \mu_1$. The stability boundary of a stationary wave which propagates in an unbounded space $p_*(L) = p(+0, L)$ has the following asymptotic forms: $p_* \approx 1/(L-1) - 0.5$ as $L \to 1+0$ and $p_* \approx 1/L^2$ as $L \to \infty$ (Fig.2). When $p < p_*(L)$, the wave is stable. We note that, in the corresponding one-dimensional problem when $L > 1$, solution (5) is stable for any p (it is, of course, necessary to factorize the perturbations with regard to neutral shifts).

Let us now consider the solution of problem (8). The eigenvalues of problem (8) form complex conjugate pairs and the constraint $\operatorname{Im}\lambda \geqslant 0$ is subsequently assumed to apply everywhere. We shall denote the solutions of problem (8) by $\varphi = (\xi, \eta, \zeta, \rho)^T$ (T is the symbol of transposition). By analogy with (6), we determine two sets of solutions of (8): $\varphi^- = (\varphi_0^-, \varphi_1^-, \varphi_2^-, \varphi_3^-)$ and $\varphi^+ = (\varphi_0^+, \varphi_1^+, \varphi_2^+, \varphi_3^+)$ which are designated by their asymptotic forms

$$\begin{aligned}
&s \to -\infty, \quad \varphi_k^- \sim (\Lambda, p_k(1+p_k) - (\lambda+\mu), \Lambda p_k,\\
&p_k^2(1+p_k) - p_k(\lambda+\mu))^T \exp(p_* s),\\
&\varphi_m^- \sim (1, 0, p_m, 0)^T \exp(p_m s)\\
&s \to +\infty, \quad \varphi_k^+ \sim (0, 1, 0, q_k)^T \exp(q_k s),\\
&\varphi_m^+ \sim (1, 0\ p_m, 0)^T \exp(p_m s)
\end{aligned} \tag{12}$$

$$(k = 0,1;\quad m = 2,3)$$
$$p_{0,1} = (-1 \mp [1 + 4L(\Lambda + \lambda + L\mu)]^{1/2})/(2L)$$
$$p_{2,3} = -{}^{1}/_{2} \pm ({}^{1}/_{4} + \lambda + \mu)^{1/2},\quad q_{0,1} = p_{0,1}\,|_{\Lambda=0}$$

(the branch of the square root with a cut along the positive part of the real semi-axis; $\sqrt{-1} = i$). Expressions (12) were obtained by solving problem (8) with the formal replacement of $\partial\Phi/\partial X$ and $\partial\Phi/\partial Y$ by the corresponding constant values as $\pm\infty$. Formulae (12) hold in the case when the characteristic equations have non-multiple roots.

The following conditions, which are obtained from (8), enable one, using the specified asymptotic forms (12), to continue φ^- and φ^+ on the whole of the s-axis:

$$[\xi] = [\eta] = 0,\quad [\zeta] = [L\rho] = -(1 + p)\,\xi_* \qquad (13)$$
$$\xi_* = \xi(s_*),\quad [f] = f(s_* + 0) - f(s_* - 0)$$

The two sets φ^- and φ^+ are linearly dependent and, therefore,

$$\varphi^- = S(\lambda)\,\varphi^+ \qquad (14)$$

where $S(\lambda)$ is a constant 4 x 4 matrix. The problem of determining the eigenvalues λ is formulated with the help of conditions on S or on the matrix P which is its inverse.

In studying the question of the possible loss of stability of solution (5), we shall confine ourselves to investigating the possibility of the appearance of eigenvalues in the right-hand half-plane $\operatorname{Re}\lambda \geqslant 0$ and, since they were previously confined to the upper half-plane, we shall subsequently assume a value of λ which belongs to the first quadrant. In the first quadrant, φ_1^-, φ_2^- are bounded as $s \to -\infty$. Hence, the eigenfunctions φ must be a linear combination of both φ_1^-, φ_2^- and φ_0^+, φ_3^+. On the other hand, it follows from (14) that

$$\varphi_j^- = S_{0j}\varphi_0^+ + S_{1j}\varphi_1^+ + S_{2j}\varphi_2^+ + S_{3j}\varphi_3^+,\quad j = 1, 2$$

and the combination $\varphi = c_1\varphi_1^- + c_2\varphi_2^-$ must not contain φ_1^+ and φ_2^+. In order for this to be so, it is necessary that the rows $(S_{11}\varphi_1^+, S_{21}\varphi_2^+)$ and $(S_{12}\varphi_1^+, S_{22}\varphi_2^+)$ should be proportional, that is,

$$\begin{vmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{vmatrix} = 0 \qquad (15)$$

Eq.(15) served to determine the eigenvalues λ which lie in the first quadrant. Calculations of the matrix S using (12) and (13) yield

$$S_{11} = [(p_1 - q_0)\,[p_1(1 + p_1) - (\lambda + \mu)] - \Lambda(1 + p)/L]\,(q_1 - q_0)^{-1}\exp[(p_1 - q_1)\,s_*] \qquad (16)$$
$$S_{12} = (1 + p)\,[L(q_0 - q_1)]^{-1}\exp[(p_2 - q_1)\,s_*]$$
$$S_{21} = \Lambda(p_1 - p_3 - p - 1)\,(p_2 - p_3)^{-1}\exp[(p_1 - p_2)\,s_*]$$
$$S_{22} = (p_2 - p_3 - p - 1)\,(p_2 - p_3)^{-1}$$

Substitution of (16) into (15) leads to the equation for

$$(p_1 - q_0)\,(p_2 - p_3 - p - 1)\,[p_1(1 + p_1) - (\lambda + \mu)] - \Lambda(1 + p)\,(p_2 - p_1)/L = 0$$

which, by making use of the identity $p_1(1 + p_1) - (\lambda + \mu) = (p_1 - p_2)\,(p_1 - p_3)$, can be abbreviated to $(p_1 - p_2)$, $(p_1 - q_0)$ and we finally obtain

$$\Delta(\lambda, p, \mu, L) \equiv (p_1 - p_3)\,(p_2 - p_3) + (1 + p)\,(p_3 + q_0 + 1/L) = 0 \qquad (17)$$

As $L \to 0$, Eq.(17) yields

$$(\Lambda + 2\lambda)\,[1 + 4(\lambda + \mu)]^{1/2} = 2\lambda(\Lambda - 1) + \Lambda - 4\mu \qquad (18)$$

By squaring (18), we obtain

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \qquad (19)$$
$$a_1 = {}^{3}/_{2}\Lambda - {}^{1}/_{4}\Lambda^2 + \mu,\quad a_2 = {}^{1}/_{2}\Lambda + (2\Lambda - 1)\,\mu,\quad a_3 = ({}^{1}/_{2}\Lambda + {}^{1}/_{4}\Lambda^2)\,\mu - \mu^2$$

The eigenvalues λ, lying in the first quadrant, are the roots of Eq.(19) which satisfy Eq.(18). We shall investigate (19) with the aid of the Hurwitz criterion. Lines specified by the equations

$$a_1 = 0,\quad a_1a_2 - a_3 = 0,\quad a_3(a_1a_2 - a_3) = 0$$

subdivide the plane of the parameters Λ and μ into regions with a different number of roots of Eq.(19) in the right-hand half-plane, λ. In the case of the regions

$$0 < \Lambda \leqslant \sigma - 1,\quad \sigma - 1 < \Lambda < 2/\sigma + 3 + \sigma,\quad 2/\sigma + 3 + \sigma \leqslant \Lambda$$
$$(\sigma = (1 + 4\mu)^{1/2},\ \Lambda > 0,\ \mu \geqslant 0)$$

the roots in the right-hand half-plane, λ, are respectively: one real root, no roots, a pair of complex roots. In the latter case, it is only the root with $\operatorname{Im}\lambda \geqslant 0$ which is of interest because Eq.(18) was obtained for the first quadrant.

It can be shown that the real root does not satisfy Eq.(18) while a root from the complex conjugate pair does satisfy it. Problem (8) also has as a solution the root which is the complex conjugate of that found. Intersection of the imaginary axis by the pair of roots under consideration is associated with loss of stability. For large values of μ, the boundary of stability

$$\Lambda(\mu) = 3 + (3 + 4\mu)(1 + 4\mu)^{-1/2} \tag{20}$$

tends asymptotically to the parabola $4\mu = (\Lambda - 3)^2 - 5$. The roots of Eq.(19) are readily calculated for the points (Λ, μ) of curve (20). One of them is $\lambda = -a_1$ and, therefore, for the pair of other roots we have

$$\lambda = \pm i\omega,\quad \omega = {}^1/_2(1 + 4\sigma + 5\sigma^2 + 2\sigma^3)^{1/2}$$
$$\sigma = (1 + 4\mu)^{1/2},\quad \mu = k^2$$

A numerical investigation was carried out for values of $0 < L < 1$. The roots ω and p were determined as a function of the parameters $\mu = k^2, L$ from the equation $2L\Delta(i\omega, p, \mu, L) = 0$ using Newton's method. The profiles of these functions for fixed values of L are shown in Figs.1 and 3.

When $L = 1$, Eq.(17) has the roots $\lambda = -\mu, \lambda = -\mu - {}^1/_4$ which do not lie in the first quadrant and solution (5) is therefore stable.

When $L > 1$, the boundary of stability is calculated in the following manner. In Eq.(17), we put $\lambda = 0$. Rearrangement of the parts of the equation and then taking its square leads to an equation which is quadratic in p

$$Q_0 p^2 + Q_1 p + Q_2 = 0 \tag{21}$$
$$Q_0 = 2(1 - L - L^2) - 8L^2\mu + 2L(L - 1)\sigma + 2(L - 1)r + 2L\sigma r$$
$$Q_1 = 4(1 - L) - 2r^2\sigma + 2(L - 4L\mu - 2)r + 2(1 + L)\sigma r$$
$$Q_2 = -8L\mu - 8L(2 - L + 2L^2)\mu^2;\quad r = (1 + 4L^2\mu)^{1/2}$$

When $\mu = 0$, this equation is identically satisfied for any p and L. Actaully, it follows from Eq.(17) that $\lambda = 0$, when $\mu = 0$, is the eigenvalue for any L, p. As in /6/, this is associated with the existence of a single parameter family of solutions of Eqs.(4) which are obtained from (5) by a shift along s. The neutral hypersurface $p = p(\mu, L)$ is obtained as the solution of Eq.(21) with a plus sign in front of the radical while the other value is negative. The eigenfunction $\varphi = S_{22}\varphi_1^- - S_{21}\varphi_2^-$ which corresponds to $\lambda = 0$ defines $\xi(s)$ and $\eta(s)$ in (11). The limiting profile $p_*(L) = p(+0, L)$ can be calculated if (21) is divided by μ and one passes to the limit $\mu \to +0$. We then obtain

$$p_*(L) = (-2L^2 + 3L + 1 + [(2L - 1)(2L^3 - 5L^2 + 8L - 1)]^{1/2}) \times [4L(L - 1)]^{-1}$$

The solution which has been found was subjected to a numerical check.

Let us now consider problem (9). We shall make use of Weil's formula for the spectrum /7/

$$\mu_n = 4\pi\,|\Omega|^{-1} n,\quad n \to \infty \tag{22}$$

(here, $|\Omega|$ is the cross-sectional area of the cylinder in dimensionless coordinates /1/). If the contour of Ω is deformed such that the normals ν are only slightly changed, then the spectrum $\{\mu_n\}$ is only slightly changed. If, however, the normals are not close when the contours Ω are close, the points of the spectrum may be moved by a finite amount. A sample of the same cross-sectional area as the initial sample and with a similar but very broken boundary can be prepared and, by virtue of the finite change in μ_n, this can lead to a finite change in the boundary of stability $\Lambda_\Omega(L)$ (see /10/). At the same time, by virtue of (22), $\Lambda_\Omega(L)$ is solely dependent on a finite number of μ_n with small numbers.

If $\mu_1 > \mu^*(L)$ (Fig.1), then $\Lambda_\Omega(L) = \Lambda^*(L)$ and a loss of stability when $0 \leqslant L < 1$ occurs as in the one-dimensional problem. In the remaining cases, it follows that one should use formula (10) to calculate $\Lambda_\Omega(L)$, only taking the finite part of the spectrum into consideration for which $\Lambda(\mu_n, L) < \Lambda^*(L)$. The condition $\mu_1(\Omega) = \mu^*(L)$ when $0 \leqslant L < 1$ and the corresponding cross-section of the cylinder from the family of such cross-sections on which it is satisfied will be referred to as the critical cross-section. All cross-sections of smaller size than the critical cross-sections belonging to the family have one and the same boundary of stability $\Lambda_\Omega(L) = \Lambda^*(L)$ and loss of stability occurs as in the one-dimensional problem. In the case of a square with a side of length, $l_c = \pi/\sqrt{\mu^*(L)}$ (Fig.4), the cross-section will be the critical cross-section while, in the case of a circle, the critical radius

$R_c \approx 0{,}5861\ l_c(L)$.

When $L > 1$, if $p < p_*(L)$, the stationary wave is stable irrespective of the shape of the cylinder. When $p > p_*(L)$, we determine the critical size of the square as $l_c = \pi/\sqrt{\mu^*(L,p)}$, where $\mu^*(L,p)$ is a transformation of the formula for the neutral hypersurface $p = p(\mu, L)$. For a specified L and $p > p_*(L)$, a stationary wave in a cylinder of square cross-section with a length of a side of the square $l < l_c$ is stable. When $l = l_c$ there is a loss of stability and, when there is a "short" perturbation, the solution corresponding to this instability is a stationary wave which is now inhomogeneous with respect to the variables x and y.

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ON THE THEORY OF THE FILTRATION OF A LIQUID IN A POROUS MEDIUM UNDER BULK HEATING BY A HIGH-FREQUENCY ELECTROMAGNETIC FIELD*

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The process of the filtration and warming up of an extremely viscous liquid (bitumen) in a porous medium where there is a bulk thermal source due to the absorption of energy from a high-frequency electromagnetic field (hfemf) is investigated. This problem is associated with the analysis of bituminous oils /1/, the filtration of which is only realized in practice after a preliminary heating of the reservoir with the help of a hfemf, for example /2-5/.

It is assumed that the bitument is initially either in the liquid (mobile) or solid (immobile) state. Under the action of the bulk thermal source, the bitumen is heated, whereupon it melts, expands, flows, and moves with respect to the immobile, solid, porous skeleton of the rock under the pressure differential which is created. A closed system of differential equations is obtained and fundamental dimensionless similarity criteria are established which characterize the above-mentioned processes. The different types of stationary or limiting solutions which are realized during stationary or sufficiently lengthy heating of the medium are studied. When they exist, these solutions may be used to estimate the effectiveness of the actual process (to estimate the limiting length of the fusion zone, the extent of heating of the liquid bitumen and the characteristic time required for the process to attain a stationary state, etc., for example) and as tests to check the correctness of the various approximate and numerical methods for solving the resulting system of non-linear differential equations.

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